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Some Recent Advances in the Investigation of Shell Buckling

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A survey is presented which includes the buckling of shells under loads for which the shell is sensitive to initial imperfections. Results for such cases show that improvements in experiment and theory have produced previously unobtainable agreement. The necessity of the correct (and consistent) theoretical specification of boundary conditions is then demonstrated. Recent stiffened cylinder results are surveyed to expose the large effects on the buckling strength of internal or external stiffening, axial load applied eccentric to the wall neutral surface, and the addition of small meridional curvature.

Nomenclature

a	= chord radius of shell
d	= stiffener spacing at midlength of shell
p	= pressure
s	= sensitivity angle, $s = 4\theta/\pi$, $-3 \lesssim s < 1$
t	= thickness of shell
$\bar{t}$	= cross-section area per unit circumference at midlength of shell
u, v, w	= displacements in the x, y, and radial-directions, respectively
x, y	= meridional and circumferential directions
D	= plate stiffness, $Et^3/12(1 - \mu^2)$
E	= Young's modulus for material
H	= meridional rise
L	= axial length of shell
N_x, N_y, N_{xy}	= stress resultants
P	= applied axial compressive force
R	= cylindrical radius at midlength of shell or radius of spherical shell
Z	= curvature parameter
θ	= angle of slope in the initial postbuckling range of characteristic load-displacement curve
μ	= Poisson's ratio
τ	= applied shear stress
$\epsilon_x, \epsilon_y, \gamma_{xy}$	= strains
∇^4	= $(\partial^4/\partial x^4) + 2(\partial^4/\partial x^2\partial y^2) + (\partial^4/\partial y^4)$

Subscripts

A	= prebuckling values
B	= buckling values
cl	= value given by conventional theory
cr	= value at buckling
$,x; ,y$	= partial differentiation of the principal symbol with respect to x and y
$(\ )'$	= differentiation with respect to x
$(\bar{\ })$	= constant applied value (displacement or force only)

Introduction

STIFFENED and unstiffened shells appear in the structure of almost every aerospace vehicle. Buckling is often the prime consideration in the design of such shells. In the last few years, major advances have been made in shell buckling knowledge. These advances were made possible mainly by the construction of near-perfect test specimens in conjunction with careful experimentation and by the use of modern computers in conjunction with more complete theory. The purpose of this paper is to identify the most important recent advances and to review them and their contributions to better design.

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A survey paper on shell structures has recently been given by Hoff,[1] and a summary of shell buckling problems concerned with imperfection sensitivity was given by Budiansky and Hutchinson.[2] In Ref. 1, a history of the use of thin shells is presented and the interaction between analysis and design, as well as future trends in shell structures, is discussed. The present survey focuses on two areas of recent endeavor which are not considered in detail in Refs. 1 and 2. First, improved theory including nonlinear shell prebuckling behavior is discussed. Second, the significance of recent experiments involving near-perfect shell specimens is indicated. The implications of these theoretical and experimental results are used to identify various subjects which the author believes to be the most important recent advances in shell buckling research. These subjects are interrelated and are discussed as two broad classes of phenomena: 1) imperfection sensitivity and 2) boundary conditions and unsymmetric shell wall geometry, both of which have a major influence on the predictability of shell buckling behavior.

Derivation of Consistent Buckling Equations for Thin Isotropic Circular Cylinder

Many of the recent improvements in shell buckling theory can be associated with the use of a consistent set of equations including nonlinear prebuckling behavior. In order to illustrate what is meant by "nonlinear shell prebuckling behavior" and how it affects the form of the buckling equation, the present derivation is given. A thin isotropic circular cylindrical shell has been chosen for simplicity but the behavior is illustrative of the more general case. In the large-deflection Donnell theory, the basic differential equations of equilibrium for a cylinder are

$$N_{x,x} + N_{xy,y} = 0 \qquad N_{y,y} + N_{xy,x} = 0$$
$$D\nabla^4 w + (N_y/R) - (N_x w_{,xx} + N_y w_{,yy} + 2N_{xy} w_{,xy}) = p \tag{1}$$

Equations (1) represent summation of the forces in the axial, circumferential, and radial directions, respectively. According to Hooke's law,

$$N_x = [Et/(1 - \mu^2)](\epsilon_x + \mu\epsilon_y)$$
$$N_y = [Et/(1 - \mu^2)](\epsilon_y + \mu\epsilon_x) \tag{2}$$
$$N_{xy} = [Et/2(1 + \mu)]\gamma_{xy}$$

The nonlinear strain-displacement relations are

$$\epsilon_x = u_{,x} + \tfrac{1}{2}w_{,x}{}^2 \qquad \epsilon_y = v_{,y} + (w/R) + \tfrac{1}{2}w_{,y}{}^2$$
$$\gamma_{xy} = u_{,y} + v_{,x} + w_{,x}w_{,y} \tag{3}$$

Equations (1–3) provide a complete set of nine equations in nine unknown stress resultants, strains and displacements which, together with the boundary conditions, specify the problem.

For problems where the loading is axisymmetric, for example, a cylinder under pressure, in torsion, or subjected to axial compression (whether loaded through the wall neutral surface or loaded eccentrically), the prebuckling deformations would be axisymmetric. If it is required that the prebuckling deformations are axisymmetric, that is, they are functions of x only, they may be obtained directly from Eqs. (1–3):

$$
\begin{gathered}
N_{xA}' = 0 \qquad N_{xyA}' = 0 \\
Dw_A'''' + (N_{yA}/R) - N_{xA}w_A'' = p \\
N_{xA} = [Et/(1-\mu^2)](\epsilon_{xA} + \mu\epsilon_{yA}) \\
N_{yA} = [Et/(1-\mu^2)](\epsilon_{yA} + \mu\epsilon_{xA}) \\
N_{xyA} = [Et/2(1+\mu)]\gamma_{xyA} \\
\epsilon_{xA} = u_A' + \tfrac{1}{2}w_A'^2 \qquad \epsilon_{yA} = w_A/R \qquad \gamma_{xyA} = v_A'
\end{gathered} \tag{4}
$$

The subscript A denotes prebuckling values, and the primes denote differentiation with respect to x.

To the prebuckling stress resultants, strains, and displacements, are added the infinitesimal changes that occur at buckling:

$$
\begin{aligned}
u &= u_A(x) + u_B(x,y) \\
&\cdots \\
\epsilon_x &= \epsilon_{xA}(x) + \epsilon_{xB}(x,y) \\
&\cdots \\
N_x &= N_{xA}(x) + N_{xB}(x,y) \\
&\cdots
\end{aligned} \tag{5}
$$

The subscript B denotes changes that occur at buckling, and, as written, these buckling deformations may be nonaxisymmetric. Thus, by considering the changes that occur at buckling to be infinitesimal, buckling is represented either by an extremum or by a bifurcation point of the characteristic load-displacement diagram. Buckling equations may now be obtained by substituting Eqs. (5) into Eqs. (1–3), by subtracting out identities (4) relating subscript A deformations, and then by neglecting terms nonlinear with respect to the infinitesimal subscript B deformations:

$$
\begin{gathered}
N_{xB,x} + N_{xyB,y} = 0 \qquad N_{yB,y} + N_{xyB,x} = 0 \\
D\nabla^4 w_B + (N_{yB}/R) - (N_{xA}w_{B,xx} + N_{xB}w_A'' + \\
N_{yA}w_{B,yy} + 2N_{xyA}w_{B,xy}) = 0 \\
N_{xB} = [Et/(1-\mu^2)](\epsilon_{xB} + \mu\epsilon_{yB}) \\
N_{yB} = [Et/(1-\mu^2)](\epsilon_{yB} + \mu\epsilon_{xB}) \\
N_{xyB} = [Et/2(1+\mu)]\gamma_{xyB} \\
\epsilon_{xB} = u_{B,x} + w_A'w_{B,x} \\
\epsilon_{yB} = v_{B,y} + (w_B/R) \\
\gamma_{xyB} = u_{B,y} + v_{B,x} + w_A'w_{B,y}
\end{gathered} \tag{6}
$$

The equations are homogeneous in the subscript B deformations. Allowing no change in applied load (or applied displacement) at buckling results in homogeneous boundary conditions and thus gives the required homogeneous system for an eigenvalue problem; the buckling load (or applied displacement) is then the eigenvalue.

In order to solve a problem, it is necessary first to solve Eqs. (4) for the prebuckling displacements and stress resultants. Equations (4) have nonlinear terms: the $N_{xA}w_A''$ term in the third equilibrium equation and the $\frac{1}{2}w_A'^2$ term in the definition of the ϵ_{xA} strain. The boundary conditions for this prebuckling solution may be consistent with those of the buckling solution. In this survey, analysis based on this type of prebuckling state will be referred to as "consistent." Most buckling analyses, however, have been based on the assumption either that linear membrane theory holds in the prebuckling range or, equivalently, that w_A equals a constant in the prebuckling range. For the cylindrical shell, either of these assumptions omits the term Dw_A'''' in the third equilibrium equation in Eq. (4) and the nonlinear terms. For these assumptions, all the w_A terms disappear from Eqs. (6). In this paper, solutions based on these types of assumptions will be termed "conventional." An intermediate assumption that may be made is that prebuckling deformations and stresses may be given by linear bending theory. Although this approach has merit, it will not be considered further in this survey since very few results are available.

Experiment and Theory for Imperfection-Sensitive Thin Isotropic Shells

For certain shell configuration and loading combinations such as the cylinder in axial compression and the sphere or toroidal segment in external pressure, drastic variations in the buckling load have been obtained experimentally for nominally identical shells. Generally, experimental buckling loads have been much lower than theoretical values. For many years, it has been speculated that these discrepancies might be attributed primarily to very small initial imperfections in shell geometry. On the other hand, substantial research has also been carried out to develop empirical buckling criteria based on energy considerations in the prebuckling and postbuckling regimes. Recent research which greatly clarifies this situation is reviewed in this section.

Cylinders in Axial Compression

Experimental results for the buckling of unstiffened cylindrical shells in axial compression have exhibited the most erratic behavior of all. Generally, the buckling load is much less than the classical load based on conventional theory which corresponds to an average stress of $0.6Et/R$. A summary of experimental and theoretical results available for the cylinder in axial compression is indicated in Fig. 1.

Experiment

Prior to 1961, experimental buckling loads for various radius-thickness ratios fell within the shaded area. Recent experimental results are represented by the various symbols in Fig. 1. Extreme care has been taken in these recent experiments to manufacture near-perfect shell specimens in order to minimize the effects of initial imperfections. Results are presented in this paper only for those specimens which the experimenters picked as "excellent" or near-perfect prior to testing.

Tennyson[3,4] describes the construction and testing in axial compression of five accurately made cylindrical shells 8 to 10 in. in diameter. His results for the four near-perfect shells are shown by the circular symbols in Fig. 1. These shells were spun-cast from photoelastic plastic ($E = 3.7 \times 10^5$ psi), and the ends were set in plaster of Paris so that uniform loading was achieved. The results for each cylinder were repeatable as the cylinders buckled elastically. The buckling load varied inversely as the percent variation of shell thickness, and for the near-perfect specimens, results very close to the theoretical results were repeatedly obtained.

Horton and Durham manufactured three nickel cylindrical shells ($E = 24 \times 10^6$ psi) by electroforming.[5] A thin coat of nickel 0.004 in. thick was deposited on an aluminum mandrel, and the coating was freed by rapid cooling of the mandrel to leave shells 8 in. long and 2.906 in. in diameter. Another mandrel placed inside, with clearance of 0.004 in.

from the cylinder wall, kept the buckling deformations elastic during testing. The edges of the cylinders were bonded with epoxy to end plates. The test results (omitting one shell of lower quality) are shown by the triangles in Fig. 1. These tests, together with others, demonstrated that the load and the position on the shell at which buckling occurs is a function of geometric imperfection and that the random distribution of imperfections which occur in cylinders is a main contributory cause to the substantial scatter that is observed in cylinder tests.

Almroth, Holmes, and Brush[6] also manufactured nickel cylindrical shells by electroforming on an aluminum mandrel. Buckling loads are available for nine 0.0055-in.-thick nominally identical cylinders 10 in. long and 5.95 in. in diameter. Of these shells, they identified only one as excellent and its buckling load is represented by the square symbol in Fig. 1. A steel mandrel was used inside the shells for testing, and the edges were held in grooves in the end plates with potting material.

Babcock and Sechler[7] plated copper ($E = 13 \times 10^6$ psi) cylindrical shells on an accurately machined wax mandrel and then melted the mandrel out of the shell. The cylinders were 8 in. in diameter, 10 in. long, and 0.0045 in. thick. Both ends of the cylinders were secured by casting with a low-melting-temperature alloy: one to a groove in an end ring and the other to a load measuring cylinder. The end ring was then cemented to the end plate of the testing machine. Their test results for essentially perfect cylinders are shown by the diamond symbols. They also found that any small departures from initial straightness lowered the buckling stress and that this effect for inward displacements was greater than for outward displacements.

The fact that very high values of the buckling load were obtained when initial irregularities were carefully minimized confirms that irregularities are the major reason for the reduction in critical load. It was also found for the very delicate shells tested that a disturbance such as a gust of air can substantially reduce the experimental buckling load.

Theory

Theoretical advances for the buckling of cylindrical shells in compression have also been in the direction of improved agreement between experiment and theory. These advances have been embodied in use of the nonlinear bending theory for prebuckling displacements and stresses and consistent prebuckling and buckling boundary conditions. The results shown in Fig. 1 by the horizontal lines are those obtained for simple support and clamped boundary conditions along with in-plane edge displacements, $u = \bar{u}$, $v = 0$ (for results for these and other boundary conditions see Refs. 8–11). These boundary conditions are believed to bracket those of the test results shown. From Fig. 1, it can be seen that consistent theory for perfect cylinders and recent experiments on near-perfect shells are in excellent agreement for values of radius-thickness ratio less than about 300 and in fairly good agreement above 300. Thus, the previous wide disparity between conventional theory and experiment has been greatly narrowed by a combination of better experiment and theory.

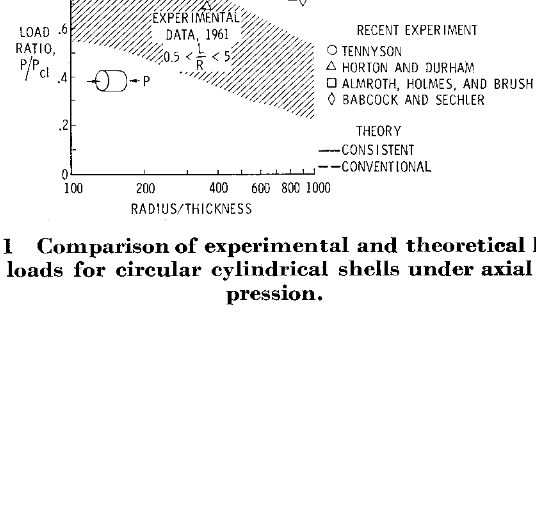

Fig. 1 Comparison of experimental and theoretical buckling loads for circular cylindrical shells under axial compression.

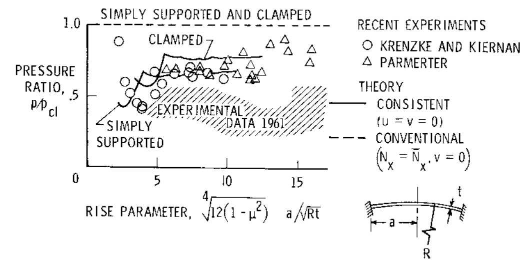

Fig. 2 Comparison of experimental and theoretical buckling pressures for shallow spherical shells under external pressure.

Shallow Spherical Shells under External Pressure

Like the cylinder in axial compression, early experiments for shallow spherical shells under external pressure gave erratic results and did not agree with theory. A summary of experimental and theoretical results now available is indicated in Fig. 2 as a function of the rise parameter.

Experiment

The experimental buckling loads available prior to 1961 are plotted within the shaded areas. Recent results[12,13] as represented by the symbols, provide significantly higher buckling loads than those obtained previously.

In Fig. 2, the results shown by the circular symbols were obtained by Krenzke and Kiernan[12] with shells accurately machined from 7075-T6 aluminum ($E = 10.8 \times 10^6$ psi) bar stock. The radius of curvature was either 2 or 3 in., the various wall thicknesses were between 0.01 and 0.04 in., and the various chord radii were between 0.4 and 1 in. The bar stock was machined so that the shell edges were supported integrally by a $\frac{1}{2}$-in.-thick annular plate. Unlike previous tests, these results follow a definite pattern much like consistent theory.

The results shown by the triangular symbols were obtained by Parmerter.[13] Copper ($E = 16 \times 10^6$ psi) electroplated on machined wax forms was used to manufacture accurate shells. The edge was set in epoxy and held in place by aluminum rings, and the shells were loaded with oil pressure. The shells were 8 in. in diameter with 20- or 40-in. radius of curvature. Unusually high values of load were obtained especially at higher values of the rise parameter.

Theory

Allowing the shallow spherical shell to compress uniformly in the prebuckling state (conventional theory) will lead to the $0.6Et/R$ stress at buckling. As for the case of the cylinder in axial compression, this assumption is inconsistent with satisfaction of the boundary conditions of a supported shell segment. Consistent nonlinear theoretical results have been obtained for symmetrical buckling with a clamped edge and $u = v = 0$ by Budiansky, Weinitschke, and Thurston[14–16] for asymmetric buckling with a clamped edge and $u = v = 0$ by Huang in Ref. 17, and for symmetric and asymmetric buckling with a simply supported or clamped edge and $u = v = 0$ by Weinitschke in Ref. 18.† The first scallop to the

† A paradox still exists in the region of $3 < [12(1 - \mu^2)]^{1/4} a/(Rt)^{1/2} < 5$ (see Fig. 2) in that the theory for clamped edges indicates lower loads than simple support. Results have not yet been obtained for simply supported edges with wave number equal to one. Possibly, a study of this case will resolve this paradox.

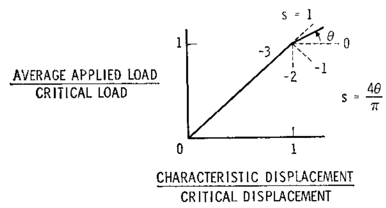

Fig. 3 Definition of initial postbuckling slope.

left for each curve represents symmetric buckling. Thus, recent very carefully conducted experiments for the shallow spherical cap under external pressure give substantially higher values of buckling load, most of which fall between the two stability curves for clamped edges and simple support. Again, much of this improved agreement can be attributed to careful limiting of initial imperfections in the test specimens, confirming further the important role of imperfection sensitivity in controlling shell behavior.

Imperfection Sensitivity

Practical shell structures are not near-perfect in geometry, and techniques must be found for determining the amount of degradation in buckling strength caused by initial imperfections. Artificial buckling criteria such as postbuckling minimum loads and loads corresponding to equal energy states do not account for imperfections and, therefore, do not appear to be necessary or desirable. Recent work on the effects of initial imperfections and sensitivity to initial imperfections is discussed in this section.

The development of methodology for imperfection sensitivity analysis has been initiated by Koiter,[19,20] but its principal impetus has been a series of research investigations by Budiansky and Hutchinson.[2] In these works, the load is expanded in a series in terms of powers of the maximum lateral buckling deflection for a given shell-loading combination. The objective is determination of the second coefficient of this series which provides a first approximation measure of sensitivity to imperfection and can be related to the strength reduction caused by geometric imperfections having the shape of the buckling mode. The approach was discussed in detail in Ref. 2, and, therefore, it will not be described further here.

A related measure of imperfection sensitivity is provided by the initial postbuckling slope of the curve of load vs characteristic displacement (whose convolution with the load yields work). This quantity is used here because it provides a simple vehicle for description of the basic concepts. The slope has been calculated in Ref. 20 and in some other past imperfection sensitivity analyses; its definition is illustrated by Fig. 3. A parameter s corresponding to the initial postbuckling slope is defined such that s will vary between -3 and 1 as the postbuckling curve varies from a condition where it doubles back on itself to a condition where it is tangent with the prebuckling curve. For s positive, imperfections will not change the buckling load appreciably; for s negative, imperfections will decrease the buckling load. Although a quantitative measure of this load reduction with a given magnitude of imperfection is not so readily achieved with s as with the Koiter parameter, s provides a more suitable basis for comparing one shell-load combination with another. Moreover, s provides a measure of the expected violence of the buckling process and hence the expected damage; for $s < -2$, "snap-through" buckling will always occur, whereas for $-2 > s > 0$, buckling will be violent or gentle depending on whether loading or characteristic displacement is controlled.

For the first two problems just described, the cylinder in axial compression and the spherical shell under external pressure, the sensitivity angle s (calculated on the basis of conventional theory) is $s = -3$. This correlates well with the known maximum sensitivity of these shell-load combinations. Of course, many shell-loading combinations are less sensitive to imperfections (see, for example, the interesting buckling and postbuckling study of the effect of initial geometric imperfections on the behavior of cylinders subjected to hydrostatic pressure by Kemper in Ref. 21). Results from another less sensitive combination are shown in Fig. 4 for cylinders in torsion. For this loading, results based on conventional theory are identical with those based on consistent theory. Initial postbuckling slopes of the characteristic torque-twist curve have been determined by Budiansky in Ref. 22. The parameter s for this loading is defined in the sketch on Fig. 4. In the upper figure, available experimental buckling torques are represented by shaded area when they plot close together and by + symbols when they are far apart; for comparison in the lower figure, corresponding sensitivity angles are shown for various values of the curvature parameter. The Koiter second coefficient, as determined in Ref. 22, shows a distribution similar to that shown for s. The experimental cylinders were clamped at the edges; simple support is almost impossible to get in the laboratory for this loading. Therefore, theoretical torque and the sensitivity angle have been calculated on the basis of clamped edges. The results indicate insensitivity at low values of Z, crossover at $Z = 10$, maximum sensitivity at $Z = 40$, and then a drop off to zero at larger values of Z. However, compared to the thickness, the probable relative size of imperfection is larger at larger values of Z and this probably accounts for the wide variation in buckling torques there. The shaded area and symbols represent more than 100 experiments, all done more than 20 years ago with no extraordinary precautions to minimize initial imperfections. If the upper two test points are excluded, it is not hard to imagine that the upper envelope of the test results follows the trend of the sensitivity curve. However, the correlation of the sensitivity angle with experience is anything but precise. Perhaps this should not be surprising since the character of the imperfections in the experimental cylinders is not accounted for in this correlation. Clearly, further efforts to improve and evaluate means to predict imperfection effects are needed.

An Appraisal of Recent Results for Imperfection-Sensitive Shells

The foregoing comparisons indicate that the reason theory and experiment did not agree in the past was mainly due to initial imperfections in the experiment and to a lesser extent to assumptions in the theory. Agreement cited between theoretical results for a perfect shell with correct boundary conditions and results from nearly perfect experimental specimens indicates the theory is accurate. With the theory established as accurate and with initial imperfections identified as the cause of disagreement, it is now important

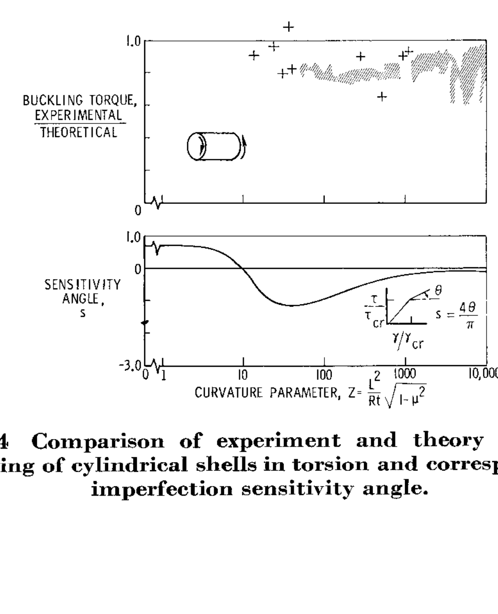

Fig. 4 Comparison of experiment and theory for the buckling of cylindrical shells in torsion and corresponding imperfection sensitivity angle.

to determine quantitatively the sensitivity to initial imperfections of shell-loading combinations of interest. By studying the theoretical initial postbuckling behavior, a rough measure of the imperfection sensitivity may be determined. Additional research in this area is needed to achieve a reliable measure of strength reduction from imperfections.

Theoretical Specification of Boundary Conditions and Wall Configuration

Although it has been pointed out in the previous sections that initial imperfections are a major cause of disagreement between theory and experiment and that determination of a consistent prebuckling state using nonlinear bending theory is often necessary for agreement, it must be emphasized that there are also other aspects of buckling analysis, especially of practical shell structures such as stiffened shells, that require careful consideration. For example, although in the past it was thought that boundary conditions have relatively little effect on the buckling load of shells except for shallow configurations, recent research indicates important differences with change in boundary conditions. Furthermore, recent analytical studies show that the detailed specification of wall geometry and load application point are important considerations in predicting the strength of stiffened shell structures. These effects are discussed below.

Support Conditions

It will be convenient first to consider the effect on buckling load of a variety of support conditions. For completeness, corresponding conventional buckling calculations will also be shown. Results from Ref. 11 for the buckling of a cylinder is combined axial compression and pressure with various edge supports are presented as an example problem. In Fig. 5, curves are shown for simply supported and for clamped cylinders with four different combinations of neutral surface edge conditions for each. The solid curves are based on consistent theory, whereas the dashed lines were found using conventional theory. It will be noticed that the conventional theory results are both above and below the consistent theory results, and for part of the range of one simply supported case $u = \bar{u}$, $N_{xy} = 0$, the conventional results are considerably below. For buckling under pressure alone ($P = 0$) various boundary conditions are seen to give sizable differences in buckling pressures. Under axial compression alone ($p = 0$) the condition of simple support in which $N_{xy} = 0$ gives buckling stresses equal to $0.3Et/R$ or only 50% of the $v = 0$ value. This boundary condition is unusual and probably does not appear in design, but it does dramatically demonstrate the strong influence of boundary conditions.

The lower buckling load was first calculated by conventional theory by Ohira in Ref. 23 for the semi-infinite cylinder

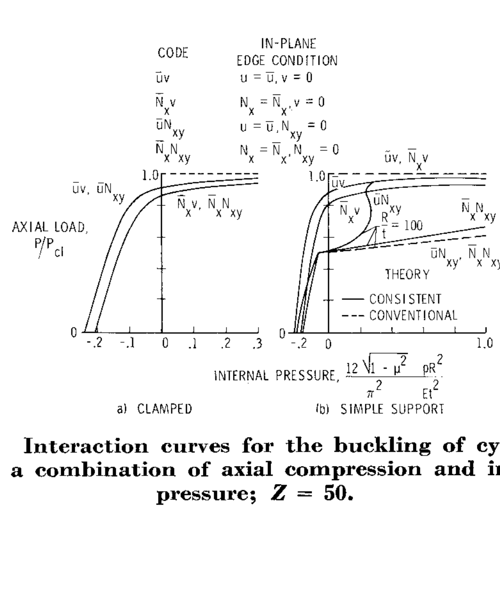

Fig. 5 Interaction curves for the buckling of cylinders under a combination of axial compression and internal pressure; $Z = 50$.

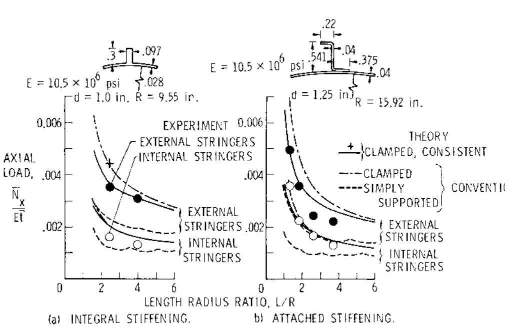

Fig. 6 Comparison of experiment and theory for the buckling of axially compressed axially stiffened cylinders.

n axial compression. Similar conventional theory results for finite cylinders were found by Hoff and Rehfield in Ref. 24. Independently, Almroth[11] also found conventional theory results for the finite cylinder (in addition to consistent theory results). The importance of various edge condition for the finite cylinder buckling under external pressure was pointed out earlier by Sobel.[25]

Eccentricity of Stiffening

Theory for shells with stiffening attached eccentrically to one side of the shell wall relates the extensional strains and the bending strains in a much more complicated way than theory for the symmetrically stiffened shell wall. This was first pointed out by van der Neut[26] in 1947, but was not generally recognized as important until tests by Card[27,28] demonstrated large effects on strength. Figure 6 presents these experimental data. The results indicate that externally stiffened cylinders have carried more than twice the load carried by corresponding internally stiffened cylindrical test data and bracketing them are conventional theory for clamped and simply supported end conditions from Ref. 28 which includes the effect of one-sided stiffening. Also indicated is consistent theory from Refs. 29–31.‡ These results show wide differences between simply supported and clamped edge results as well as between inside and outside stiffening. As expected, the test results are generally closer to the theoretical results for clamped ends. Note that external stiffening is not always best. For some other stiffening configurations, quite different buckling load results are obtained, and there appears to be no simple rule to determine which stiffening is the most efficient for a desired case without a complete calculation. Many papers have been written recently on eccentrically stiffened shells using conventional theory. In particular, the comprehensive paper by Singer, Baruch, and Harari[32] on the stiffened cylinder under axial compression should be mentioned.

Eccentricity of Loading

Often the edges of a shell are loaded through surfaces other than the neutral surface such as the locus of centroids of the stiffeners or through the middle surface of the skin. Moments due to such load eccentricity cause unusual prebuckling deformations and stresses, and consistent theory is required. Theoretical results obtained from Refs. 29–31 are shown in Fig. 7 for two sets of externally and internally stiffened cylinders and one set of cones loaded axially through eccentric simple supports. The results indicate that eccentricity of loading may cause significant variations in the buckling load of stiffened cylindrical and conical shells. For these con-

‡ Note that a discrepancy exists between the calculated results for externally stiffened cylinders in Refs. 29 and 30 (the solid curve in Fig. 6a) and a single calculation reported in Ref. 31 (indicated by the + symbol). Both calculations were made with essentially the same theory, and there is no obvious reason for the discrepancy.

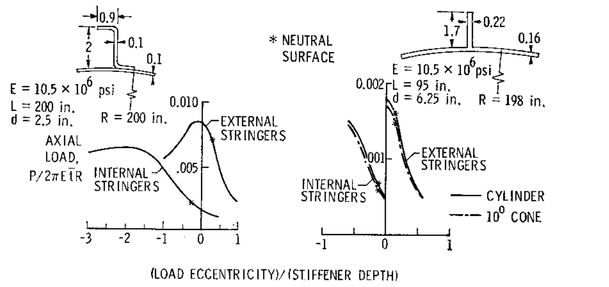

Fig. 7 Influence of eccentrically applied compressive loads (measured from skin center) on the buckling loads of simply supported axially stiffened cylinders and cones.

figurations, up to four times higher buckling loads are available with load applied on the inside of the neutral surface for both internal and external stiffening compared to the cases for loading at the neutral surface. Again there are considerable differences in load in internal and external stiffening, but there is little difference in results between the cylinder and the 10° cone.

Addition of Small Meridional Curvature to a Stiffened Cylinder

In addition to the changes in geometric or loading details, large effects on strength of a stiffened cylindrical shell may result from small changes in the gross geometry. For example, results which have been calculated to study the effects of the addition of small meridional curvature in a stiffened cylinder in axial compression have been interpolated to get the curves presented in Fig. 8. The plot on the left in Fig. 8 presents conventional theory results due to McElman in Ref. 33 and from Card and Jones in Ref. 28, where the cylinder values are the calculated results for one of the test cylinders in Fig. 6. For internal stiffening, the addition of 7.8% outward rise to the meridian was found to increase the load by the factor 9. The plot on the right in Fig. 8 presents results obtained from consistent theory by Almroth, Bushnell, and Sobel.[31] The cylindrical configuration in the plot on the right is the same as the cylinder loaded through the neutral surface in Fig. 7. Here, increases of the order of 7 times were found with addition of 6% outward rise. Generally, higher loads may be obtained with outward rise and with internal stiffening. The differences in buckling load between internal and external stiffening are not as large for the doubly curved shells as they are for the cylinder and cone. The results also indicate a large decrease in axial load-carrying capacity with the addition of small negative meridional curvature. Note that this behavior for stiffened shells with the addition of small positive meridional curvature does not have a direct counterpart in unstiffened shells. Unstiffened shells in compression do not carry appreciably more load with the addition of small meridional curvature.[7]

An Appraisal of Wall Configuration and Boundary Condition Effects

The examples presented point out the need for the use of correct and consistent boundary conditions in shell buckling predictions. The use of linear membrane prebuckling theory may give buckling loads which are conservative or unconservative compared to consistent theory and the use of correct boundary conditions including account of eccentric loading may well be the difference between agreement and disagreement with experiment. Account must also be taken of any eccentricity of stiffening in stiffened shells.

Based on examples presented using current theory, it appears that large increases in load-carrying ability may be afforded 1) by location of one-sided stiffening with demonstrated gains of 200%, 2) by applying the load eccentrically with respect to the neutral surface with demonstrated gains of 400%, and 3) by the addition to stiffened cylinders of small positive meridional curvature with demonstrated gains of 700%. Of course, the spectacular gains indicated by theory are not additive and should be tempered by other considerations including imperfections, plasticity, and the possibility of other loadings such as tension or bending becoming critical instead of compression. In addition, these large gains probably will not be realized for optimum design. Expected gains due to load eccentricity and the addition of positive meridional curvature have not yet been verified experimentally.

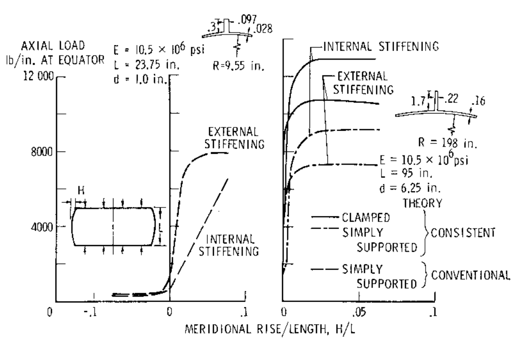

Fig. 8 Compressive buckling loads for axisymmetric doubly curved axially stiffened shells.

To study the effects that have been discussed for desired configurations by means of consistent theory requires extensive and complex computations. A computer program that is generally available to make such calculations is described by Almroth, Bushnell, and Sobel in refs. 31, 34, and 35 for the asymmetric buckling of shallow shells of revolution under axial compression and pressure with various edge conditions including eccentric loading and ring support. This program includes the effects of nonlinear (bending) prebuckling deformations and stresses. Results from this program are in agreement with other theoretical results and in good agreement with available experiments.

Concluding Remarks

Results of recent experimental and theoretical research in buckling of shell structures show apparent agreement between the results of well-founded theory and carefully prepared and conducted experiments. These results suggest that disagreement of theory with experiment in the past was mainly due to initial imperfections of the experimental shells and, to a lesser extent, to simplifications of the theory. However, other strong influences on shell-buckling behavior have also been uncovered with the aid of modern computers in conjunction with more complete theory; these include large effects on load-carrying ability of eccentricity of stiffening, eccentricity of load, and of the addition to stiffened cylinders of slight meridional curvature.

It can be concluded from these advances that properly posed buckling theory is correct and there is no need for artificial buckling criteria such as the arbitrary use of postbuckling minimums or "equal energy criteria." Moreover, the importance of initial imperfection is verified, and a beginning has been made on the crucial problem of quantitatively evaluating the effect of imperfections in a design calculation. Finally, it is concluded that boundary conditions must be specified precisely and small details in geometry and load path must be taken into account in many design calculations.

Note: See the comment by L. J. Hart-Smith and the reply by the author, on p. 2459 and p. 2460 of this issue.

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Technical Comments

Comments on "Some Recent Advances in the Investigation of Shell Buckling"

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IN Ref. 1, Stein has concluded that the classical thin-shell buckling analyses are basically correct. Confining attention here to the case of cylinders under pure axial compression, his Fig. 1 indicates that the classical theory predicts the same buckling stress whether the ends be completely clamped ($u - \bar{u} = 0$, $v = 0$, $w = 0$, $\partial^2 w/\partial x^2 = 0$) or incompletely restrained ($u - \bar{u} \neq 0$, $v = 0$, $w = 0$, $\partial^2 w/\partial x^2 = 0$). He then compares this theory with only those experiments involving clamped ends. This raises two questions. First, if the theory applies for both end conditions, why should not the experimental results for cylinders with incompletely restrained ends also agree with the theory? Stein, himself,

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(Continued on next page)

concludes from the total experimental evidence that the end conditions are of vital importance. Second, does the theory really correspond to the situation in the experiments quoted by Stein? The classical theory for the circumferential ring-type buckling,[2] does apply to either completely clamped or incompletely restrained end conditions. However, for the classical solution for the diamond buckling mode,[2] it is necessary [Ref. 2, Eq. (f), p. 464] that the diamond wrinkles extend to the ends of the cylinder, at which $u - \bar{u} = \pm A \sin n\theta \neq 0$. Therefore, the classical solution for the diamond buckling mode is not comparable with any of the experimental evidence quoted by Stein on the diamond buckling mode with completely clamped end conditions. All of his references (4–7) specifically indicate the diamond buckling mode and such severe end clamping as to completely suppress all displacements there. Furthermore, the high-speed movie-camera observations recorded in his Refs. 4 and 6 clearly show that the buckling mode observed was the diamond mode right from the onset of buckling. There is no evidence cited in any of his relevant references (4–7) of any change of buckling mode from the rings to the diamonds.

Therefore, the problem of reconciling the classical equilibrium-bifurcation analysis with the comparable experimental evidence appears to remain unsolved. Consequently, it appears to be premature to conclude that the classical shell-buckling analyses are basically correct.

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Reply by Author to L. J. Hart-Smith

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This reply is based on the following interpretation of certain of the expressions used by Mr. Hart-Smith (see the preceding comment). The expression "classical analysis" in the comments is interpreted to mean the same as the expression "conventional analysis" in the author's paper.[4] "Clamped" in the comments seems to refer to one type of in-surface restraint ($u = \bar{u}$, $v = 0$), whereas in the author's paper, it referred to a common type of out-of-surface restraint ($w = \partial w/\partial x = 0$). Also, evidently, "diamond buckling results" in the comments refers to conventional results based on the sine-sine configuration [given by Eq. (f), p. 464 of Ref. 2 of the comments) rather than postbuckling results corresponding to the diamond configuration that is commonly seen in experiment.

In as much as Mr. Hart-Smith's comments are directed primarily to the results presented in Fig. 1 of Ref. 4, it is appropriate to review in detail the author's interpretation of those results. Figure 1 indicates that if $v = w = 0$ at the edges, conventional theory predicts the same average buckling stress for the cylinder in axial compression no matter whether the axial displacement or the axial force is held constant at the edges or whether the edge slope or moment is zero. Consistent theory, on the other hand, indicates a variety of buckling loads depending on the edge restraint all within about 20% of the corresponding conventional theory value (see also Fig. 4, $p = 0$). All available near-perfect experimental results were plotted in Fig. 1. In all the experiments, attempts were made to obtain clamped edges ($w = \partial w/\partial x = 0$) with $u = \bar{u}$, $v = 0$; therefore, the experiments should be compared to the consistent theory for the same boundary conditions—the uppermost solid line in Fig. 1. Failure to achieve complete edge restraint would lead to lower experimental buckling loads and may contribute a small portion of the differences between experiment and this theoretical line which are shown in Fig. 1.

The equilibrium-bifurcation analysis for shell buckling has been strongly supported in the author's paper based on the agreement in Fig. 1 between theory and experiment for the buckling load. Mr. Hart-Smith questions the strong support on the basis of his interpretation of comparisons of buckling configurations obtained by experiment with that predicted by theory. This criticism deserves discussion. There has long been some misunderstanding as to what configuration should be seen at buckling. Consistent theory indicates that the configuration that should be expected from a perfect cylinder at buckling is a combination of the axisymmetric prebuckling configuration and the asymmetric initial buckling configuration (not to be confused with the final "diamond-shaped" postbuckling configuration commonly observed in the laboratory). Indication that the question may be resolved appears in Ref. 1 in which evidence has now been obtained through experiments on geometrically near-perfect cylinders that the shape at buckling is indeed a combination of an axisymmetric configuration and the asymmetric buckling configuration.

It is not clear to the author how to interpret Mr. Hart-Smith's comments with regard to "change of buckling mode from the rings to the diamonds." If the Comments refer to change from axisymmetric prebuckling deformations to asymmetric buckling configuration, then it should be remarked that these effects are accounted for in consistent theory and the prebuckling deformations have been observed experimentally.[2] If the Comments refer to change in buckle pattern with change in shell dimensions, then it should be noted that from consistent calculations made so far for the cylinder in axial compression, the critical buckling loads have always corresponded to $n \neq 0$ (see, for example, Table II of Ref. 3) and, therefore, no ring buckling has been indicated. Note that ring buckling loads for consistent theory represent asymptote values rather than bifurcation values.

Mr. Hart-Smith's statement that the classical sine-sine solution is not precisely comparable to experiment because the in-surface boundary condition on u is not satisfied is, strictly speaking, correct. However, this point is irrelevant to the author's comparison of consistent theory with experiment; moreover, the demonstrated insensitivity of both conventional and consistent solutions (Fig. 4) to change in this boundary condition suggest that comparison of even the conventional solutions with laboratory experiments should not be condemned on this basis. In summary:

1) Agreement in buckling load should be obtainable between experiment and theory for the same boundary conditions. It has been obtained for the near-perfect clamped cylinder in axial compression.

2) New evidence has been found to indicate it is possible to get the theoretical buckling configuration experimentally for this loading. The axisymmetric prebuckling configuration has been observed experimentally, but the axisymmetric buckling configuration will probably not appear in thin cylinders buckling elastically in pure axial compression.

3) These results and others cited in the author's paper for this and other loadings all point to the conclusion that equilibrium-bifurcation shell-buckling analysis is basically correct.

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